Exercise 25

Water is leaking out of an inverted conical tank at a rate of 10,000 cm^3/min at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the height of the water is 2 m, find the rate at which water is being pumped into the tank.

Solution

According to the law of conservation of mass, matter is neither created nor destroyed. This means that what goes in the cone must come out, or else matter will accumulate inside of it.

$$\text{Rate of mass accumulation} = \text{Rate of mass in} - \text{Rate of mass out}$$

$$\frac{dm}{dt} = \frac{dm_{\text{in}}}{dt} - \frac{dm_{\text{out}}}{dt}$$

Use the fact that mass is density times volume.

$$\frac{d(\rho V)}{dt} = \frac{d(\rho V_{\text{in}})}{dt} - \frac{d(\rho V_{\text{out}})}{dt}$$

The density of water is constant, so it can be pulled in front of each derivative.

$$\rho\frac{dV}{dt} = \rho\frac{dV_{\text{in}}}{dt} - \rho\frac{dV_{\text{out}}}{dt}$$

Divide both sides by ρ.

$$\frac{dV}{dt} = \frac{dV_{\text{in}}}{dt} - \frac{dV_{\text{out}}}{dt}$$

Water is leaking out of the tank at a rate of 10,000 cm^3/min.

$$\frac{dV}{dt} = \frac{dV_{\text{in}}}{dt} - 10\,000$$

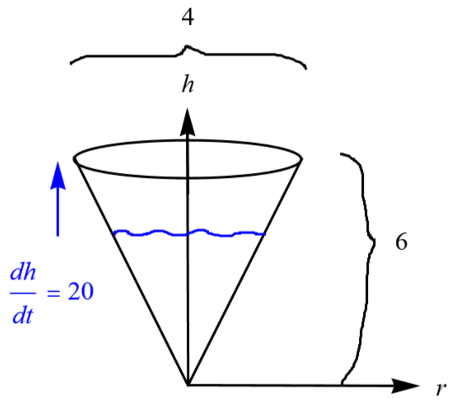
Solve for the rate of water entering the cone.

$$\frac{dV_{\text{in}}}{dt} = 10\,000 + \frac{dV}{dt}$$

The volume of a cone is $V = (1/3)\pi r^2 h$.

$$\begin{aligned}\frac{dV_{\text{in}}}{dt} &= 10\,000 + \frac{d}{dt}\left(\frac{1}{3}\pi r^2 h\right)\\ &= 10\,000 + \frac{\pi}{3}\frac{d}{dt}(r^2 h) \qquad (1)\end{aligned}$$

Use the given geometry of the cone to write h in terms of r.



The cone's vertex is at the origin, so the equation of the line representing the cone's edge is

$$h = \frac{\text{rise}}{\text{run}} r = \frac{6}{2} r = 3r.$$

Since h and dh/dt are given, solve this equation for r,

$$r = \frac{h}{3},$$

and plug it into equation (1).

$$\begin{aligned}
\frac{dV_{\text{in}}}{dt} &= 10\,000 + \frac{\pi}{3}\frac{d}{dt}(r^2 h) \qquad (1) \\
&= 10\,000 + \frac{\pi}{3}\frac{d}{dt}\left[\left(\frac{h}{3}\right)^2 h\right] \\
&= 10\,000 + \frac{\pi}{3}\frac{d}{dt}\left(\frac{h^3}{9}\right) \\
&= 10\,000 + \frac{\pi}{27}\frac{d}{dt}(h^3) \\
&= 10\,000 + \frac{\pi}{27}(3h^2) \cdot \frac{dh}{dt} \\
&= 10\,000 + \frac{\pi h^2}{9} \cdot (20) \\
&= 10\,000 + \frac{20\pi h^2}{9}
\end{aligned}$$

Therefore, when the height of the water is 2 meters (200 centimeters), the rate at which water is being pumped into the tank is

$$\left.\frac{dV_{\text{in}}}{dt}\right|_{h=200} = 10\,000 + \frac{20\pi(200)^2}{9} \approx 289\,253\ \frac{\text{cm}^3}{\text{min}}.$$